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Notices of the Aëronautical Society of Great Britain.

The Activities of the Society.

Arrangements are now being made to resume the activities of the Society in the New Year by holding meetings, extending the Library, and continuing its educational work in London and the provinces. Full details will be announced in due course.

Resignation of the Secretary.

The present Secretary, being unable to give his whole time to the work, is relinquishing the post in the New Year and devoting himself to Munition work. Arrangements are now being made to appoint his successor, whose name will be announced in due course.

Assistant Secretary.

Miss Phyllis Boyd has been appointed Assistant Secretary from 1st January, 1917.

The Aeronautical Institute of Great Britain.

Evidence frequently reaches the Secretary of confusion existing between the identity of this Society and the above mentioned body (which was formed in September, 1915) in the minds of the public. It is hoped that members will lose no opportunity of clearing up such confusion when they are aware of its existence, by pointing out that this Society (which is the recognised authority on the scientific and technological aspects of Aeronautics) has not delegated any of its functions

XUM

to any other body. Members will appreciate the inadvisability of dissipating the force of the Aeronautical movement in this country by the multiplication of societies.

Death of Sir George White and Sir Hiram Maxim.

The following minutes were passed at the last meeting of the Council:—

The Council of the Aëronautical Society of Great Britain has heard with great regret of the death of its late distinguished member, Sir George White, Bart., who joined the Society in 1911, and to whose foresight, energy and perseverance Aeronautics will always remain deeply indebted. The Council desire to express their sincere sympathy with his family in their great loss.

The Council of the Aëronautical Society of Great Britain has heard with great regret of the death of its late distinguished member, Sir Hiram Stevens Maxim, who joined the Society in 1891, and was for many years a member of its Council.

His great energy and mechanical genius were applied to Aeronautics at a time when flight was still only an aspiration and fully earned for him his place among the great aeronautical pioneers. The Council desire to extend to his family the deepest sympathy in their great loss.

THE LONGITUDINAL INITIAL MOTION AND FORCED OSCILLATIONS OF A DISTURBED AEROPLANE.

BY S. BRODETSKY, M.A., LECTURER IN APPLIED MATHEMATICS IN THE UNIVERSITY OF BRISTOL.

WITH AN INTRODUCTION BY PROFESSOR G. H. BRYAN, SC.D., F.R.S., BANGOR.

INTRODUCTION.

The present investigation fills a gap in the applications of the formal methods of rigid dynamics to the study of the motions of aeroplanes.

The principal methods for the *approximate* solution of dynamical problems together with their aeronautical applications may perhaps be conveniently classified as follows:—

Rigid Dynamics.	Aeronautics.
1. Small oscillations about a state of steady motion and applications to stability.	The longitudinal and lateral inherent stability of an aeroplane.
2. Steady motion in a circle.	Steering of aeroplanes.
3. Initial and impulsive changes of motion under given forces.	Effect of sudden wind disturbances on an aeroplane.

It will be seen that this order of development, which was contemplated by me long before the realisation of artificial flight, was that required at that time to answer the questions, whether an aeroplane could fly without upsetting, how it could be steered in a curve, and how it could be adapted for use in boisterous weather.

In this paper the methods of initial and impulsive motions are applied to studying the effects of permanent and transitory changes of wind velocity on the motion of an aeroplane, in the two-dimensional problem in which the wind velocity has no lateral component. The method of forced oscillations is subsequently applied to the discussion, under similar conditions, of the effects of periodic wind fluctuations after the free oscillations have died out.

It might be thought that the effects of atmospheric disturbances could be studied by applications of the theory of small oscillations, in all cases in which the change of wind velocity is small in comparison with the relative velocity of the aeroplane. As a matter of fact, any such attempt when initial conditions are taken into account, would probably lead to intractable calculations of the most hopeless character. Considering only the longitudinal problem, and adopting the notation of *Stability in Aviation*, the small added velocity components and angular displacement u, v, θ, are each of the form

$$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t} + C_4 e^{\lambda_4 t}$$

where λ_1, λ_2, λ_3, λ_4, are roots of the period equation. Assuming this equation expressed in the form of a determinant, the coefficients C for the three components are proportional to the minors of the determinant, and would be sufficiently complicated even if the values of λ were known. But when initial conditions are taken into account, the difficulty would not stop there, but the

final values of the coefficients would again have to be evaluated in the form of new determinants.

The problem is, however, essentially adapted for solution by the method of initial motions. When a sudden wind strikes an aeroplane, of course the most important point is to ascertain its initial effects on the behaviour of the machine, because after a short time it may be safely left to the pilot to adapt the course to the altered conditions. It therefore becomes sufficient to express the solutions in a series of ascending powers of the time, and for most purposes only a few coefficients will probably be required. As, however, the problem can now be solved by successive approximations, it will become merely a question of straightforward drudgery to carry the approximations further than Mr. Brodetsky has done, should this be desirable at some future time.

Although the problems of the permanent and the impulsive disturbances are closely related to each other, there is a certain important difference between them.

If we consider a permanent change of wind velocity whose components are u and v, the motion of the aeroplane *relative to the air* is exactly the same as if, instead, the aeroplane had velocity components equal and opposite to u and v suddenly impressed on it. It would thus appear to be easier to study the relative motion under the new conditions than the actual motion. Unfortunately this would not give the information most needed, namely, the change of motion which the aeroplane would experience under the circumstances.

In the case of the "impulsive disturbance" which strikes the aeroplane for a very short time and as suddenly dies down, the effect will be to produce initial changes in the velocity components of the aeroplane. The first thing is to obtain expressions for these changes. The study of the subsequent motion will be exactly analogous to the problem of *relative* motion due to the permanent disturbance.

The study of the initial and other effects of *lateral* disturbances is bound to be not less interesting than the problem of the present paper, and I hope that Mr. Brodetsky will be able to deal with it. There are, of course, further possible developments, such as applications to systems specified by the resistance derivatives of the force and couple components, also attempts to deal in a more or less satisfactory way with the difficulties that are bound to arise in cases of large disturbances, where the velocity of the disturbing current is comparable with the speed of the aeroplane.

But I entirely agree with Mr. Brodetsky that the best course is to start with the simpler problems, and thus lead up by gradual stages to the more difficult ones.

G. H. Bryan.

The Problems Considered.

(1) Suppose that an aeroplane is moving uniformly in a horizontal straight line with velocity V_0 relatively to the air, and that the air has communicated to it some additional motion. To find the subsequent motion of the aeroplane exactly seems to be beyond the scope of mathematical analysis at present. We may, however, apply the method of small oscillations in some cases if the velocity communicated to the air is small, or we may discover the initial effect to a moderate degree of approximation by solving in powers of the time measured from the beginning of the disturbance.

E. B. Wilson (*Theory of an Aeroplane Encountering Gusts, Proc. Nat. Acad.*

of Sciences, U.S.A., May, 1916, pp. 294-7) has discussed the effect on an aeroplane of a particular type of gust, which can be expressed in the form

$$J(1 - e^{-rt}),$$

where J is an intensity factor, t is the time, and r is a constant defining the time variation of the gust.

In this paper we discuss three distinct problems of disturbance of an aeroplane by an alteration in the air conditions:—

I.—The initial effect of a permanent change in the relative velocity of the air.

II.—The initial effect of an impulsive gust, of great intensity but acting only for a very short time.

III.—The forced oscillations produced by a periodic gust, undamped or damped; and the special case when the complex exponent representing the damped periodic gust is a solution of the period equation of the free oscillations of the aeroplane.

In each case a discussion of the results is given with a view to finding the type of machine that suffers least disturbance, and that rights itself as quickly as possible. These conditions are summarised at the end of the paper.

It will be seen that the object of the present investigation is contained in Nos. 4 and 13 of the *Problems* suggested in Ch. XI. of Bryan's *Stability in Aviation*. It was at Professor Bryan's suggestion that I undertook the work, and I take this opportunity of expressing my appreciation of his invaluable encouragement and help.

Two-Dimensional Motions Only Dealt With.

(2) The disturbances discussed in this paper are taken to be longitudinal, and the aeroplane is supposed to possess a plane of symmetry—the vertical plane of motion. The relative added velocity of the air has components u and v, taken to be positive when vertically upwards and horizontally backwards respectively. We do not discuss here the case of turbulent motion of the air. The aeroplane is taken to consist of two surfaces, of which the front surface—the main plane—supports the whole of the weight of the machine, whose centre of mass is at the centre of pressure of the main plane, and the rear surface acts as a neutral tail. The results can be extended to more complicated machines by means of the invariant method developed in Ch. VI. of *Stability in Aviation*. The general case of any number of variously oriented and inclined surfaces, and with three dimensional disturbances, is reserved for future investigation. The analysis is bound to be heavy, and the work of the present discussion will be helpful in the consideration of the unrestricted problem.

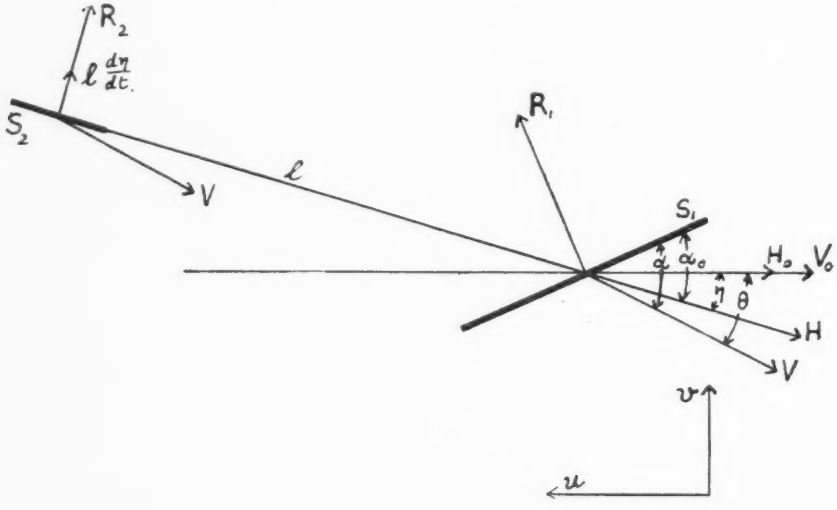
Our notation will be as follows:—

S_1 is the area of the main plane,
K_1 its coefficient of resistance;
S_2 is the area of the tail,
K_2 its coefficient of resistance;
M is the mass of the aeroplane,
Mg its weight, using dynamical units;
I is the moment of inertia of the aeroplane about an axis through the centre of mass, *i.e.*, the centre of pressure of the main plane, perpendicular to the plane of motion;
H_0, V_0, α_0 are the airscrew thrust, velocity, and angle of attack of the main plane in the undisturbed motion; the tail is tangential in the steady motion.

We adopt the hypothesis of "narrow planes at small angles," so that we may assume the air resistance to be proportional to the sine of the angle of attack, and we may neglect the rotary derivatives and the shifts of the centres of pressure.

We have the following conditions of steady motion:—

$$\left.\begin{aligned} H_0 &= K_1 S_1 V_0^2 \sin^2 \alpha_0, \\ Mg &= K_1 S_1 V_0^2 \sin \alpha_0 \cos \alpha_0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$



(3) Let the figure represent the position of the disturbed aeroplane at any instant. The main plane makes an angle α_0 with the line joining the centres of pressure of the two planes, this line making an angle η with the forward horizontal, measured downwards; the velocity V makes an angle α with the direction of the main plane. The angle θ between the velocity V and the forward horizon, measured downwards, is

$$\theta = \eta + \alpha - \alpha_0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The distance between the centres of pressure of the two planes is l.

The airscrew thrust makes a constant angle α_0 with the main plane. There is no easy theoretical means of calculating the thrust for a disturbed aeroplane flying under the influence of airscrews driven by a given engine. The best course is to assume the results of actual measurements performed with some standard type of airscrew used in the construction of aeroplanes. As the result of such measurements made at the National Physical Laboratory, and kindly communicated to me by Messrs. L. Bairstow and A. Fage, we assume

$$\frac{1}{H}\frac{\partial H}{\partial V_p} = -.011, \quad . \quad . \quad . \quad . \quad (3)$$

where V_p is the relative velocity of the air along the axis of the airscrew, measured in miles per hour. We therefore take

$$H \propto e^{-\kappa V_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

κ being a small constant, about 1/90 for velocities measured in miles per hour. Hence

$$H = H_0 e^{-\kappa (V_p - V_0)} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will often be sufficient to use the formula

$$H = H_0 \{ 1 - \kappa (V_p - V_0) \} \quad . \quad . \quad . \quad . \quad (6)$$

I.—PERMANENT CHANGE OF RELATIVE VELOCITY OF THE AIR.

(4) Let the wind have components of velocity

u horizontally backwards,
v vertically upwards.

The velocity of the front plane relatively to the air is given by

$$V_1^2 = V^2 + u^2 + v^2 + 2uV \cos\theta + 2vV \sin\theta,$$

the component normal to the plane being

$$V \sin\alpha + u \sin(\alpha_0 - \eta) + v \cos(\alpha_0 - \eta).$$

Thus the air pressure on the main plane is

$$R_1 = K_1 S_1 V_1 \{ V \sin\alpha + u \sin(\alpha_0 - \eta) + v \cos(\alpha_0 - \eta) \} \quad . \quad . \quad (7)$$

For the tail we have

$$V_2^2 = V^2 + u^2 + v^2 + 2uV \cos\theta + 2vV \sin\theta + l^2 (d\eta/dt)^2 - 2l\, d\eta/dt \{ V \sin(\alpha - \alpha_0) - u \sin\eta + v \cos\eta \}$$

the normal component being

$$V \sin(\alpha - \alpha_0) - l\, d\eta/dt - u \sin\eta + v \cos\eta,$$

and the air pressure

$$R_2 = K_2 S_2 V_2 \{ V \sin(\alpha - \alpha_0) - l\, d\eta/dt - u \sin\eta + v \cos\eta \} \quad . \quad (8)$$

Also the relative velocity of the airscrew axis is

$$V_p = V \cos(\alpha - \alpha_0) + u \cos\eta + v \sin\eta,$$

so that the airscrew thrust is

$$H = H_0 e^{-\kappa \{ V \cos(\alpha - \alpha_0) + u \cos\eta + v \sin\eta - V_0 \}} \quad . \quad . \quad (9)$$

The equations of motion in the vertical plane of symmetry are as follows:—

$$\left.\begin{aligned} M\, dV/dt &= Mg \sin\theta + H \cos(\alpha - \alpha_0) - R_1 \sin\alpha - R_2 \sin(\alpha - \alpha_0), \\ MV\, d\theta/dt &= Mg \cos\theta - H \sin(\alpha - \alpha_0) - R_1 \cos\alpha - R_2 \cos(\alpha - \alpha_0), \\ I\, d^2\eta/dt^2 &= lR_2. \end{aligned}\right\} \quad . \quad (10)$$

Initially we have

$$V = V_0,\ \alpha = \alpha_0,\ \theta = 0,\ \eta = 0,\ d\eta/dt = 0. \quad . \quad . \quad . \quad . \quad (11)$$

(5) At the beginning of the disturbance the equations (10) give

$$\begin{aligned} M\, dV/dt &= H - R_1 \sin\alpha_0, \\ MV\, d\theta/dt &= Mg - R_1 \cos\alpha_0 - R_2, \\ I\, d^2\eta/dt^2 &= lR_2, \end{aligned}$$

XUM

so that we get initially

$$\left.\begin{aligned} M(V - V_0) &= (H - R_1 \sin \alpha_0)\, t, \\ MV\theta &= (Mg - R_1 \cos \alpha_0 - R_2)\, t, \\ I\eta &= \tfrac{1}{2} l R_2 t^2. \end{aligned}\right\} \quad . \quad . \quad (12)$$

By substituting these approximations in the equations of motion (10), we can obtain a second approximation, and the process can be repeated indefinitely. At this stage, however, it is more convenient to assume V, θ, α, η, expanded in powers of the time, measured from the beginning of the disturbance, and having regard to the initial conditions (11), we therefore put

$$\left.\begin{aligned} V - V_0 &= a_1 t + a_2 t^2/2! + \ldots, \\ \eta &= b_2 t^2/2! + b_3 t^3/3! + \ldots, \\ \alpha - \alpha_0 &= c_1 t + c_2 t^2/2! + \ldots, \\ \text{so that} \quad \theta &= c_1 t + (b_2 + c_2)\, t^2/2! + \ldots \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

We shall obtain the first two coefficients in each expansion by means of substituting from (13) in the equations of motion (10), and equating coefficients of successive powers of t in each of the equations of motion. There is, of course, no limit to the number of coefficients that can be calculated in this way, but after the first two the algebra is very heavy.

As we are seeking only two coefficients in (13), we need to equate only two sets of coefficients in each equation of motion, namely, terms independent of the time, and the coefficients of the first power of the time. We need therefore to expand the quantities that enter into the equations only as far as the first power of t. It will be convenient to use the following symbols:—

$(H)_0$, $(R_1)_0$, $(R_2)_0$ are the constant terms in H, R_1, R_2;
$(H)_1$, $(R_1)_1$, $(R_2)_1$ are the coefficients of t in H, R_1, R_2.

It will also save much trouble if we at once expand in powers of t those trigonometrical functions that occur in the equations of motion. To the first power of the time we may use

$$\left.\begin{aligned} \sin\theta = c_1 t, \quad \cos\theta = 1, \\ \sin\eta = 0, \quad \cos\eta = 1, \\ \sin(\alpha - \alpha_0) = c_1 t, \quad \cos(\alpha - \alpha_0) = 1, \\ \sin\alpha = \sin\alpha_0 + c_1 t \cos\alpha_0, \quad \cos\alpha = \cos\alpha_0 - c_1 t \sin\alpha_0, \\ \sin(\alpha_0 - \eta) = \sin\alpha_0, \quad \cos(\alpha_0 - \eta) = \cos\alpha_0. \end{aligned}\right\} \quad . \quad (14)$$

(6) Equating the constant terms in (10), using (13, 14), we get

$$\left.\begin{aligned} Ma_1 &= (H)_0 - \sin\alpha_0\, (R_1)_0, \\ MV_0 c_1 &= Mg - \cos\alpha_0\, (R_1)_0 - (R_2)_0, \\ Ib_2 &= l\,(R_2)_0. \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

The coefficients of the first power of t give

$$\begin{aligned} Ma_2 &= Mgc_1 + (H)_1 - c_1 \cos\alpha_0\, (R_1)_0 - \sin\alpha_0\, (R_1)_1 - c_1\, (R_2)_0, \\ MV_0(b_2 + c_2) + Ma_1 c_1 &= -c_1\, (H)_0 + c_1 \sin\alpha_0\, (R_1)_0 - \cos\alpha_0\, (R_1)_1 - (R_2)_1, \\ Ib_3 &= l\,(R_2)_1. \end{aligned}$$

By means of (15) these equations can be reduced to

$$\left.\begin{aligned} Ma_2 &= MV_0c_1^2 + (H)_1 - \sin\alpha_0\,(R_1)_1, \\ MV_0\,(b_2 + c_2) &= -2Ma_1c_1 - \cos\alpha_0\,(R_1)_1 - (R_2)_1, \\ Ib_3 &= l\,(R_2)_1. \end{aligned}\right\} \quad . \qquad (16)$$

To the first power of t, (9) gives

$$H = H_0e^{-\kappa u}\,(1 - \kappa a_1 t),$$

so that

$$\left.\begin{aligned} (H)_0 &= H_0e^{-\kappa u}, \\ (H)_1 &= -\kappa a_1 H_0 e^{-\kappa u} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (17)$$

Equation (7) gives to the first power of t,

$$R_1 = K_1S_1\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}\left\{(V_0+u)\sin\alpha_0 + v\cos\alpha_0\right\}$$
$$\left\{1 + t\left[\frac{a_1\,(V_0+u) + c_1vV_0}{(V_0+u)^2+v^2} + \frac{a_1\sin\alpha_0 + c_1V_0\cos\alpha_0}{(V_0+u)\sin\alpha_0 + v\cos\alpha_0}\right]\right\}.$$

Hence

$$\left.\begin{aligned} (R_1)_0 &= K_1S_1\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}\left\{(V_0+u)\sin\alpha_0 + v\cos\alpha_0\right\}, \\ (R_1)_1 &= \frac{K_1S_1}{\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}}\left[\left\{a_1\,(V_0+u) + c_1vV_0\right\}\left\{(V_0+u)\sin\alpha_0 + v\cos\alpha_0\right\}\right. \\ &\qquad \left. + (a_1\sin\alpha_0 + c_1V_0\cos\alpha_0)\left\{(V_0+u)^2+v^2\right\}\right]. \end{aligned}\right\} \qquad (18)$$

Finally, from (8) we get

$$R_2 = K_2S_2v\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}\left\{1 + t\left[\frac{a_1\,(V_0+u) + v\,(V_0c_1 - lb_2)}{(V_0+u)^2+v^2} + \frac{V_0c_1 - lb_2}{v}\right]\right\}.$$

Hence

$$\left.\begin{aligned} (R_2)_0 &= K_2S_2v\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}, \\ (R_2)_1 &= \frac{K_2S_2}{\left\{(V_0+u)^2+v^2\right\}^{\frac{1}{2}}}\left[v\left\{a_1\,(V_0+u) + v\,(V_0c_1 - lb_2)\right\}\right. \\ &\qquad \left. + (V_0c_1 - lb_2)\left\{(V_0+u)^2+v^2\right\}\right]. \end{aligned}\right\} \quad . \qquad (19)$$

It is seen that $(H)_0$, $(R_1)_0$, $(R_2)_0$, do not involve any of the unknown quantities a_1, a_2, b_2, b_3, c_1, c_2. Thus the equations (15) enable us to calculate a_1, c_1, b_2 respectively. We can substitute these values in the expressions for $(H)_1$, $(R_1)_1$, $(R_2)_1$ and then the equations (16) give us a_2, c_2, b_3. The values thus obtained hold for all values of u and v. We do not calculate them formally; in what follows we shall use the equations to find them as the necessity arises.

The "Least Disturbed" Type of Aeroplane.

(7) In order to investigate the type of aeroplane that gives the least disturbances, it is convenient to assume u, v so small that their squares and products may be neglected. It might be thought that with small values of u, v, we can use the method of small oscillations. But, as is pointed out by Professor Bryan in his Introduction, this is not convenient.

In practice it is important that the path of the aeroplane shall not be too curved, in other words, it is necessary to make the curvature—$d\theta/ds$—as small as possible. Now

$$\frac{d\theta}{ds} = \frac{d\theta}{dt} \Big/ \frac{ds}{dt}$$

$$= \frac{c_1 + (b_2 + c_2)\,t + \ldots}{V_0 + a_1 t + \ldots}$$

$$= \frac{c_1}{V_0}\left\{1 + \left(\frac{b_2 + c_2}{c_1} - \frac{a_1}{V_0}\right)t\right\} \quad . \quad . \quad . \quad (20)$$

to the first power of t. Thus the machine should be so constructed that the function (20) shall be as small as possible for some time after the beginning of the disturbance. The initial curvature is c_1/V_0.

If u, v are so small that squares and products may be neglected, we get from (15), (18), (19) and (1):

$$MV_0c_1 = -\{K_1S_1V_0(2u\sin\alpha_0 + v\cos\alpha_0)\cos\alpha_0 + K_2S_2vV_0$$

Hence

$$\frac{c_1}{V_0} = -\frac{1}{V_0}\left\{\frac{2K_1S_1\sin\alpha_0\cos\alpha_0}{M}u + \frac{K_1S_1\cos^2\alpha_0 + K_2S_2}{M}v\right\} \quad . \quad . \quad (21)$$

We deduce that $1/V_0$, $2K_1S_1\sin\alpha_0\cos\alpha_0/M$ and $(K_1S_1\cos^2\alpha_0 + K_2S_2)/M$ should be as small as possible, in order to prevent too great a curvature when u and v happen to be of the same sign. Put into words, the conditions are that *the angle of attack, and the ratio of the total area to the total mass, shall both be as small as possible; the velocity should be as large as possible.*

The negative sign in (21) means that for u and v both positive—a head wind—the aeroplane rises initially, the concavity of the path being upwards, and the initial direction horizontal.

(8) The initial curvature having been found, it is important to see how it begins to change. From (15) we get

$$Ma_1 = H_0(1 - \kappa u) - K_1S_1V_0\sin\alpha_0\{(V_0 + 2u)\sin\alpha_0 + v\cos\alpha_0\}$$

$$= -H_0\left\{\left(\frac{2}{V_0} + \kappa\right)u + \frac{v}{V_0}\cot\alpha_0\right\} \quad . \quad . \quad . \quad (22)$$

by (1). Thus a_1 is small. Also c_1 is small by (21). Hence we can write

$$(R_1)_1 = K_1S_1V_0(2a_1\sin\alpha_0 + c_1V_0\cos\alpha_0) \quad . \quad . \quad . \quad . \quad (23)$$

Further, by (15),

$$b_2 = lK_2S_2vV_0/I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence b_2 is also small, so that we can write

$$(R_2)_1 = -K_2S_2V_0(lb_2 - c_1V_0) \quad . \quad . \quad . \quad (25)$$

Hence we get from (16)

$$MV_0(b_2+c_2) = -\cos\alpha_0(R_1)_1 - (R_2)_1$$
$$= -K_1S_1V_0\cos\alpha_0(2a_1\sin\alpha_0 + c_1V_0\cos\alpha_0) - K_2S_2V_0(c_1V_0 - lb_2) \quad (26)$$

It is thus seen that $(b_2+c_2)/c_1$ is finite, whilst a_1/V_0 is small. Hence we need not consider the term a_1/V_0 in (20). Also, if we take $v=0$ to simplify the algebra, we get

$$MV_0(b_2+c_2) = \frac{2(2+\kappa V_0)K_1S_1H_0\sin\alpha_0\cos\alpha_0}{M}u + \frac{2(K_1S_1\cos^2\alpha_0 + K_2S_2)K_1S_1V_0^2\sin\alpha_0\cos\alpha_0}{M}u,$$

so that

$$\frac{b_2+c_2}{V_0} = \frac{2K_1S_1\sin\alpha_0\cos\alpha_0}{M}\left\{\frac{K_1S_1\cos^2\alpha_0 + K_2S_2}{M} + \frac{(2+\kappa V_0)K_1S_1\sin^2\alpha_0}{M}\right\}u. \quad . \quad (27)$$

In order that the curvature shall diminish quickly, we find that, independently of the sign of u, the numerical value of the quantity (27) must be as large as possible. Thus *the "recovery" of the curvature gives conditions that conflict with the conditions for small initial curvature.* A compromise has to be effected. Remembering that in any aeroplane the angle of attack is small, it follows that the term in (27) involving V_0 is of little importance, especially as it is also multiplied by the small constant κ. Hence *it is safest to make the velocity as large as possible, whilst the angle of attack, the ratio tail/main plane, and the ratio total area/load, should be small, consistent with a safe margin for stability.* (A similar case of conflicting conditions when "recovery" is considered, is to be found in a paper by Bryan and Brodetsky on "Long Tails versus Short Tails in Aerodynamics," AËRONAUTICAL JOURNAL, April-June, 1916.)

Similar results hold when $u=0$.

(9) It is also of interest to consider the angular motion of the aeroplane. From (13) we get

$$d^2\eta/dt^2 = b_2 + b_3t \quad . \quad . \quad . \quad . \quad . \quad (28)$$

For u, v small we have

$$b_2 = lK_2S_2V_0v/I, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

to the first order of small quantities. Thus for η the condition is that b_2 shall be small, *i.e.*,

$$lK_2S_2V_0/I \text{ small} \quad . \quad . \quad . \quad . \quad (30)$$

consistent with a safe margin of stability.

The quantity b_3 limits this condition.

II.—IMPULSIVE GUST.

(10) If we suppose that there is a momentary gust of great violence in the plane of motion, and then a return to the former air conditions, the problem is modified. The effect of the gust is to give certain impulsive pressures on the

XUM

planes constituting the aeroplane, thus producing changes in the velocity and in the rotation. These we calculate as follows:—

Using the general notation of Part I., let us assume that during a very short time there is a large additional motion of the air, represented by components u^1, v^1, horizontally backwards and vertically upwards respectively. During the impulsive action the orientation of the aeroplane remains unaltered. Thus we get on the two planes the impulsive pressures

$$\left.\begin{aligned} &K_1S_1\int(u^{1^2}+v^{1^2})^{1/2}\,(u^1\sin\alpha_0+v^1\cos\alpha_0)\,dt,\\ &K_2S_2\int(u^{1^2}+v^{1^2})^{1/2}\,v^1dt, \end{aligned}\right\} \quad . \quad . \quad (31)$$

the integration being over the short interval during which the gust acts. The velocity of the aeroplane itself will produce no appreciable effect, and the airscrew will also have no effect during the impulse, since its thrust is small for a large relative velocity, and the time of action is short.

It is reasonable to assume that the impulsive wind is constant in direction, except for some extraordinary cyclone about a horizontal axis. We may then put

$$u^1=v^1\cot\beta, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

β being some constant angle. Thus we get that the total backward horizontal impulse, the total upward vertical impulse, and the total clockwise angular impulse, are respectively

$$\left.\begin{aligned} &K_1S_1\sin\alpha_0\,.\,\sin(\alpha_0+\beta)\,.\int(u^{1^2}+v^{1^2})\,dt,\\ &\{K_1S_1\cos\alpha_0\,.\,\sin(\alpha_0+\beta)+K_2S_2\sin\beta\}\,.\int(u^{1^2}+v^{1^2})\,dt,\\ &lK_2S_2\sin\beta\,.\int(u^{1^2}+v^{1^2})\,dt. \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

Thus the effect of the impulse is to produce on the aeroplane a horizontal backward velocity, a vertical upward velocity, and a clockwise angular velocity, of respective magnitudes $-u$, $-v$, $+\omega$, where

$$\left.\begin{aligned} u&=-\frac{K_1S_1}{M}\sin\alpha_0\,.\,\sin(\alpha_0+\beta)\,.\int(u^{1^2}+v^{1^2})\,dt,\\ v&=-\left\{\frac{K_1S_1}{M}\cos\alpha_0\,.\,\sin(\alpha_0+\beta)+\frac{K_2S_2}{M}\sin\beta\right\}\,.\int(u^{1^2}+v^{1^2})\,dt,\\ \omega&=\frac{lK_2S_2}{I}\sin\beta\,.\int(u^{1^2}+v^{1^2})\,dt. \end{aligned}\right\} \quad (34)$$

The reason for this rather artificial notation will soon be evident. For if the wind now dies down, so that we have once more a quiescent medium, the problem is essentially the same as that of the aeroplane moving with its original velocity V_0 horizontally, but with an angular velocity ω, the air being suddenly endowed

with velocities u, v, as in Part I. Thus the problem is a slight modification of that discussed in Part I, the only difference being that $d\eta/dt$ is now ω initially, not zero.

(11) If the gust is horizontal, $\beta=0$ and therefore $\omega=0$; the problem then reduces to that of Part I. In the general case when β is not zero, the expansions (13) become

$$\left.\begin{aligned} V-V_0&=a_1t+a_2t^2/2!+\ \ldots\ ,\\ \eta&=\omega t+b_2t^2/2!+b_3t^3/3!+\ \ldots\ ,\\ \alpha-\alpha_0&=c_1t+c_2t^2/2!+\ \ldots\ ,\\ \theta&=(\omega+c_1)\,t+(b_2+c_2)\,t^2/2!+\ \ldots \end{aligned}\right\} \quad . \quad . \quad (35)$$

Instead of (14) we now have

$$\left.\begin{aligned} \sin\theta=(\omega+c_1)\,t,\ \ \cos\theta&=1,\\ \sin\eta=\omega t,\ \ \cos\eta&=1,\\ \sin(\alpha-\alpha_0)=c_1t,\ \ \cos(\alpha-\alpha_0)&=1,\\ \sin\alpha=\sin\alpha_0+c_1t\ \cos\alpha_0,\ \ \cos\alpha&=\cos\alpha_0-c_1t\sin\alpha_0,\\ \sin(\alpha_0-\eta)=\sin\alpha_0-\omega t\ \cos\alpha_0,\ \ \cos(\alpha_0-\eta)&=\cos\alpha_0+\omega t\ \sin\alpha_0. \end{aligned}\right\} \quad (36)$$

The results (15) and (16) become respectively

$$\left.\begin{aligned} Ma_1&=(H)_0-\sin\alpha_0\ \ (R_1)_0,\\ MV_0\,(\omega+c_1)&=Mg-\cos\alpha_0\,(R_1)_0-(R_2)_0,\\ Ib_2&=l\,(R_2)_0; \end{aligned}\right\} \quad . \quad . \quad (37)$$

and

$$\left.\begin{aligned} Ma_2&=M\omega g+MV_0c_1\,(\omega+c_1)+(H)_1-\sin\alpha_0\ \ (R_1)_1,\\ MV_0\,(b_2+c_2)&=-M\omega a_1-2Ma_1c_1-\cos\alpha_0\ \ (R_1)_1-(R_2)_1,\\ Ib_3&=l\,(R_2)_1. \end{aligned}\right\} \quad (38)$$

We also get

$$\left.\begin{aligned} (H)_0&=H_0\ .\ e^{-\kappa u}\\ (H)_1&=-\kappa\,(a_1+\omega v)\,H_0\ .\ e^{-\kappa u} \end{aligned}\right\} \quad . \quad (39)$$

$$\left.\begin{aligned} (R_1)_0&=K_1S_1\left\{(V_0+u)^2+v^2\right\}^{1/2}\left\{(V_0+u)\sin\alpha_0+v\cos\alpha_0\right\},\\ (R_1)_1&=\frac{K_1S_1}{\left\{(V_0+u)^2+v^2\right\}^{1/2}}\left[\left\{a_1\,(V_0+u)+vV_0\,(\omega+c_1)\right\}\right.\\ &\qquad\left\{(V_0+u)\sin\alpha_0+v\cos\alpha_0\right\}\\ &\quad\left.+\left\{(a_1+\omega v)\sin\alpha_0+(c_1V_0-\omega u)\cos\alpha_0\right\}\left\{(V_0+u)^2+v^2\right\}\right]; \end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned} (R_2)_0&=K_2S_2\,(v-l\omega)\left\{(V_0+u)^2+(v-l\omega)^2\right\}^{1/2},\\ (R_2)_1&=\frac{K_2S_2}{\left\{(V_0+u)^2+(v-l\omega)^2\right\}^{1/2}}\left[(v-l\omega)\left\{a_1\,(V_0+u)+(\omega+c_1)\,vV_0\right.\right.\\ &\qquad\left.-lb_2\,(v-l\omega)-l\omega\,(c_1V_0-\omega u)\right\}\\ &\quad\left.-(lb_2-V_0c_1+\omega u)\left\{(V_0+u)^2+(v-l\omega)^2\right\}\right]. \end{aligned}\right\} \quad (41)$$

We again have all the data required to calculate a_1, a_2, c_1, c_2, b_2, b_3.

(12) If we take the case where u, v, and therefore also ω, are small, so that squares and products may be neglected, we get for the initial curvature

$$\frac{\omega + c_1}{V_0} = -\frac{1}{V_0}\left\{\frac{2K_1S_1 \sin \alpha_0 \cos \alpha_0}{M} u + \frac{K_1S_1 \cos^2 \alpha_0 + K_2S_2}{M} v - \frac{lK_2S_2}{M} \omega\right\} \quad (42)$$

It follows that the conditions for small initial curvature are the same as those found in Part I., art. 7. The results of arts. 8 and 9 are also found to hold in the case of the impulsive gust.

III.—PERIODIC GUST.

(13) We now proceed to discuss the forced oscillations produced by a periodic gust whose intensity is so small that its square may be neglected. Such a gust can be split up into a number of harmonic parts, so that we need to consider only the disturbance produced by velocity components u, v, given to the air, u, v, being small quantities, and harmonic functions of the time, possibly damped. Using the method of small oscillations, we write

$$V = V_0 + V^1, \ \alpha = \alpha_0 + \alpha^1, \ \theta = \alpha^1 + \eta, \quad . \quad . \quad . \quad (43)$$

where V^1, α^1, η, are also small quantities of the first order.

By (9) the airscrew thrust becomes

$$H = H_0\{1 - \kappa(V^1 + u)\} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Also the air pressures become

$$\left.\begin{aligned} R_1 &= K_1S_1V_0\{V_0 \sin \alpha_0 + 2(V^1 + u) \sin \alpha_0 + (v + \alpha^1V_0) \cos \alpha_0\}, \\ R_2 &= K_2S_2V_0\{(v + \alpha^1V_0) - l\, d\eta/dt\}. \end{aligned}\right\} \quad . \quad (45)$$

The equations of motion (10) give

$$\left.\begin{aligned} M\, dV^1/dt &= Mg(\alpha^1 + \eta) + H_0\{1 - \kappa(V^1 + u)\} - K_1S_1V_0 \sin \alpha_0\{V_0 \sin \alpha_0 \\ &\quad + 2(V^1 + u) \sin \alpha_0 + (v + \alpha^1V_0) \cos \alpha_0\} - \alpha^1 K_1S_1V_0^2 \sin \alpha_0 \cos \alpha_0, \\ MV_0(d\eta/dt + d\alpha^1/dt) &= Mg - \alpha^1H_0 - K_1S_1V_0 \cos \alpha_0\{V_0 \sin \alpha_0 \\ &\quad + 2(V^1 + u) \sin \alpha_0 + (v + \alpha^1V_0) \cos \alpha_0\} + \alpha^1K_1S_1V_0^2 \sin^2 \alpha_0 \\ &\quad - K_2S_2V_0\{(v + \alpha^1V_0) - l\, d\eta/dt\}, \\ I\, d^2\eta/dt^2 &= lK_2S_2V_0\{(v + \alpha^1V_0) - l\, d\eta/dt\}. \end{aligned}\right\} \quad (46)$$

Using the conditions of the steady motion, (1), we get

$$\left.\begin{aligned} M\frac{dV^1}{dt} &= Mg\eta - \frac{Mg}{V_0}(v + \alpha^1V_0) - \frac{H_0}{V_0}(2 + \kappa V_0)(V^1 + u), \\ MV_0\left\{\frac{d\eta}{dt} + \frac{d\alpha^1}{dt}\right\} &= -\frac{2Mg}{V_0}(V^1 + u) \\ &\quad - (K_1S_1 \cos^2 \alpha_0 + K_2S_2)\, V_0(v + \alpha^1V_0) + lK_2S_2V_0\, d\eta/dt, \\ I\, d^2\eta/dt^2 &= lK_2S_2V_0\{(v + \alpha^1V_0) - l\, d\eta/dt\}. \end{aligned}\right\} \quad . \quad . \quad (47)$$

(14) To simplify the algebra, we introduce the symbols

$$V^1 + u = \chi, \ v + \alpha^1V_0 = \psi. \quad . \quad . \quad . \quad . \quad (48)$$

The equations of motion then become

$$\left.\begin{aligned}
&\left\{D+\frac{H_0}{MV_0}(2+\kappa V_0)\right\}\chi- && g\eta+ && \frac{g}{V_0}\psi=Du,\\
&\frac{2g}{V_0}\chi-V_0\left(\frac{lK_2S_2}{M}-1\right)D\eta+ && && \\
&\qquad\left\{D+\frac{V_0}{M}(K_1S_1\cos^2\alpha_0+K_2S_2)\right\}\psi=Dv,\\
&-\left\{\frac{I}{lK_2S_2V_0}D^2+lD\right\}\eta+ && && \psi=0,
\end{aligned}\right\} \qquad (49)$$

using

$$D=d/dt \qquad . \quad . \quad . \quad . \quad . \qquad (50)$$

Eliminate χ, ψ; we get for η the differential equation

$$\left.\begin{aligned}
&\left[\left(\frac{I}{lK_2S_2V_0}D^2+lD\right)\left\{D+\frac{H_0}{MV_0}(2+\kappa V_0)\right\}\left\{D+\frac{V_0}{M}(K_1S_1\cos^2\alpha_0+K_2S_2)\right\}\right.\\
&-\frac{2g^2}{V_0^2}\left(\frac{I}{lK_2S_2V_0}D^2+lD\right)-V_0\left(\frac{lK_2S_2}{M}-1\right)\left\{D^2+\frac{H_0}{MV_0}(2+\kappa V_0)D\right\}\\
&\left.+\frac{2g^2}{V_0}\right]\eta=\left\{D+\frac{H_0}{MV_0}(2+\kappa V_0)\right\}Dv-\frac{2g}{V_0}Du.
\end{aligned}\right\} \qquad (51)$$

The forced oscillations in η are given by the particular integral of the equation (51). The important quantity is θ. To obtain the forced oscillations in θ we note that

$$\theta=\alpha^1+\eta$$

and that by (48)

$$\begin{aligned}
\alpha^1&=(\psi-v)/V_0\\
&=\left(\frac{I}{lK_2S_2V_0^2}D^2+\frac{l}{V_0}D\right)\eta-\frac{v}{V_0},
\end{aligned}$$

by the third equation in (49). Hence we get

$$\theta=\left(\frac{I}{lK_2S_2V_0^2}D^2+\frac{l}{V_0}D+1\right)\eta-\frac{v}{V_0} \qquad . \quad . \quad . \quad . \qquad (52)$$

It is convenient to write the operator on the left hand side of (51) in the form

$$\mathbf{A}_0D^4+\mathbf{B}_0D^3+\mathbf{C}_0D^2+\mathbf{D}_0D+\mathbf{E}_0 \qquad . \quad . \qquad (53)$$

We assume that the aeroplane possesses inherent longitudinal stability; this is true of all modern machines.

SPECIAL CASE (a).—UNDAMPED PERIODIC GUST.

(15) In this case we use the typical values

$$\left.\begin{aligned} u &= u_0 \sin (pt+\epsilon), \\ v &= v_0 \sin (pt+\epsilon). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

We get for the forced oscillations in η

$$\frac{v_0 D^2 + \left\{\dfrac{H_0}{MV_0}(2+\kappa V_0)\, v_0 - \dfrac{2g}{V_0} u_0\right\} D}{\mathfrak{A}_0 D^4 + \mathfrak{B}_0 D^3 + \mathfrak{C}_0 D^2 + \mathfrak{D}_0 D + \mathfrak{E}_0} \sin (pt+\epsilon) \quad . \quad . \quad . \quad (55)$$

The consideration of the general case introduces excessive complication. We shall then take the two extreme cases of p small and of p large, *i.e.*, of a long and of a short period.

When the gust period is very long, p is very small, and as an approximation we get from (55):

$$\eta = \frac{p}{g}\left\{\tfrac{1}{2}\frac{H_0}{Mg}(2+\kappa V_0)\, v_0 - u_0\right\} \cos (pt+\epsilon), \quad . \quad . \quad . \quad (56)$$

since by (51)

$$\mathfrak{E}_0 = 2g^2/V_0.$$

Using (1) we have for the forced oscillations:

$$\left.\begin{aligned} \eta &= \frac{p}{g}\left\{\tfrac{1}{2}(2+\kappa V_0)\tan\alpha_0 \;.\; v_0 - u_0\right\}\cos(pt+\epsilon), \\ \theta &= \frac{p}{g}\left\{\tfrac{1}{2}(2+\kappa V_0)\tan\alpha_0 \;.\; v_0 - u_0\right\}\cos(pt+\epsilon) - \frac{v_0}{V_0}\sin(pt+\epsilon). \end{aligned}\right\} \quad (57)$$

Remembering that in any aeroplane $\tan\alpha_0$ is a comparatively small quantity, whilst $\frac{1}{2}(2+\kappa V_0)$ is generally about $1\frac{1}{2}$, we see that *the important thing is to keep V_0 as large as possible*. This makes θ small; η cannot become considerable.

If the gust period is very short, p is very large, and we get as an approximation:

$$\eta = -\frac{v_0 l K_2 S_2 V_0}{I p^2} \sin (pt+\epsilon), \quad . \quad . \quad . \quad (58)$$

using the value of $\mathfrak{A}_0$ given by (51). Thus there is no danger of η becoming large. If we use (58) in (52), we get θ zero. It is therefore necessary to use a nearer approximation. Doing so we get

$$\theta = \frac{1}{p}\cos(pt+\epsilon)\left\{\frac{\mathfrak{B}_0 v_0 + \mathfrak{A}_0\left\{\dfrac{H_0}{MV_0}(2+\kappa V_0)\, v_0 - \dfrac{2g}{V_0} u_0\right\}}{\mathfrak{A}_0 V_0} - \frac{l v_0}{\mathfrak{A}_0 V_0}\right\}$$

Using the values of $\mathfrak{A}_0$ and of $\mathfrak{B}_0/\mathfrak{A}_0$ out of (51), we deduce

$$\theta = \frac{1}{p}\cos(pt+\epsilon)\left\{\frac{2K_1 S_1 \sin\alpha_0 \cos\alpha_0}{M} u_0 + \frac{K_1 S_1 \cos^2\alpha_0 + K_2 S_2}{M} v_0\right\} \quad . \quad (59)$$

Thus the forced oscillation in θ cannot become great, as the large quantity p occurs in the denominator. In any case, we see that *it is useful to make the angle of attack and the ratio area/load small.*

Combining the results for p small and for p large, we find that *it is an advantage to have a large velocity, a small angle of attack, and a small ratio area/load.* This agrees with the conditions for small initial curvature in Parts I. and II.

(16) It must be remembered, however, that *we must not get too near the limit of stability.* For if the function

$$\mathfrak{H}_0 = \mathfrak{B}_0\mathfrak{C}_0\mathfrak{D}_0 - \mathfrak{A}_0\mathfrak{D}_0^2 - \mathfrak{E}_0\mathfrak{B}_0^2 \quad . \quad . \quad . \quad . \quad (60)$$

is too near zero, the denominator in η and therefore also in θ, can become small for some values of p. In particular, if $\mathfrak{H}_0$ is actually zero, we see by (55) that for

$$p = (\mathfrak{D}_0/\mathfrak{B}_0)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

we get a factor t in η; this means that η increases indefinitely and therefore also θ. See also arts. 18 and 19.

SPECIAL CASE (b).—DAMPED PERIODIC GUST.

(17) For such a gust we use

$$\left.\begin{aligned} u &= u_0 \,.\, e^{-qt} \,.\, \sin(pt+\epsilon), \\ v &= v_0 \,.\, e^{-qt} \,.\, \sin(pt+\epsilon), \end{aligned}\right\} \quad . \quad . \quad . \quad (62)$$

where q is some positive quantity. The forced oscillation in η is now given by

$$\eta = e^{-qt} \frac{v_0(D-q)^2 + \left\{\dfrac{H_0}{MV_0}(2+\kappa V_0)\,v_0 - \dfrac{2g}{V_0}u_0\right\}(D-q)}{\mathfrak{A}_0(D-q)^4 + \mathfrak{B}_0(D-q)^3 + \mathfrak{C}_0(D-q)^2 + \mathfrak{D}_0(D-q) + \mathfrak{E}_0} \sin(pt+\epsilon). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

We again consider the extreme cases of p small and of p large. If p is very small we get approximately

$$\eta = e^{-qt} \frac{v_0 q^2 - \left\{\dfrac{H_0}{MV_0}(2+\kappa V_0)\,v_0 - \dfrac{2g}{V_0}u_0\right\}q}{\mathfrak{A}_0 q^4 - \mathfrak{B}_0 q^3 + \mathfrak{C}_0 q^2 - \mathfrak{D}_0 q + \mathfrak{E}_0} \sin(pt+\epsilon).$$

For slow damping, *i.e.*, q small, we get

$$\left.\begin{aligned} \eta &= -\frac{q}{g}\left\{\tfrac{1}{2}(2+\kappa V_0)\tan\alpha_0 \,.\, v_0 - u_0\right\} e^{-qt} \,.\, \sin(pt+\epsilon), \\ \theta &= -\left[\frac{q}{g}\left\{\tfrac{1}{2}(2+\kappa V_0)\tan\alpha_0 \,.\, v_0 - u_0\right\} + \frac{v_0}{V_0}\right] e^{-qt} \,.\, \sin(pt+\epsilon). \end{aligned}\right\} \quad . \quad (64)$$

Hence, once again, *it is advisable to have a large velocity.*

XUM

If the damping is quick, so that q is large, we get after some reduction:

$$\theta = \frac{1}{q} e^{-qt} \cdot \sin(pt+\epsilon) \left\{ \frac{2K_1S_1 \sin \alpha_0 \cos \alpha_0}{M} u_0 + \frac{K_1S_1 \cos^2 \alpha_0 + K_2S_2}{M} v_0 \right\}, \quad (65)$$

giving again the conditions that *the angle of attack and the ratio area/load should be small.*

If p is very large, we have for all values of q:

$$\theta = \frac{1}{p} e^{-qt} \cdot \cos(pt+\epsilon) \left\{ \frac{2K_1S_1 \sin \alpha_0 \cos \alpha_0}{M} u_0 + \frac{K_1S_1 \cos^2 \alpha_0 + K_2S_2}{M} v_0 \right\}, \quad (66)$$

giving the same conditions as from (65).

SPECIAL CASE (c).—DAMPED PERIODIC GUST OF THE SAME COMPLEX PERIOD AS ONE OF THE FREE PERIODS.

(18) When the gust is periodic and damped, the important special case arises, in which the complex exponent representing the time variation of the gust, agrees with one of the complex exponents representing the free oscillation periods of the aeroplane. Such a possibility exists in all stable aeroplanes. It gives a factor t in the forced oscillations, and it therefore needs special investigation. Analytically it means that $-q \pm ip$ are solutions of the algebraic equation

$$\mathfrak{A}_0 D^4 + \mathfrak{B}_0 D^3 + \mathfrak{C}_0 D^2 + \mathfrak{D}_0 D + \mathfrak{E}_0 = 0.$$

The algebra is very long if we try to work out the general result. This is, however, quite unnecessary. We know *a priori* that

$$\left.\begin{aligned} \eta &= P_1 e^{-qt} \cdot t \cdot \sin(pt+\epsilon+Q_1), \\ \theta &= P_2 e^{-qt} \cdot t \cdot \sin(pt+\epsilon+Q_2), \end{aligned}\right\} \quad . \quad . \quad . \quad (67)$$

leaving out the terms that can be supposed absorbed in the complementary functions, and which diminish with the time. The amplitudes of η, θ, are each proportional to

$$a = t e^{-qt} \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Now

$$da/dt = (1 - qt) e^{-qt},$$

$$d^2a/dt^2 = q(qt - 2) e^{-qt}.$$

Hence, a is a maximum at $t = 1/q$, and its value at the maximum is $1/eq$. The smaller q, the greater the maximum amplitude, and the later it occurs. When q is zero, the maximum value of a is infinite, and it occurs after an infinite time.

(19) It is thus seen that the danger arises from small values of q. This does actually occur in any stable aeroplane, which has a small value of the angle of attack. (See Bryan, *Stability in Aviation*, pp. 80, 84-7.) We have very approximately

$$q = \frac{\mathfrak{C}_0 \mathfrak{D}_0 - \mathfrak{E}_0 \mathfrak{B}_0}{2\mathfrak{C}_0^2}, \quad p = \left(\frac{\mathfrak{E}_0}{\mathfrak{C}_0} \right)^{1/2}, \quad . \quad . \quad (69)$$

and this makes q and p^2 both of the same order as $\tan \alpha_0$. In particular, Wilson (v.s. art. 1) describes a machine in which q is about $1/16$, so that a in (68) can be as much as 6.

We have then to investigate the case when the complex period of the gust is the same as that of the *slow* free oscillation of the aeroplane. We find that in this case the forced part of η can be taken to be approximately

$$\frac{te^{-qt} \cdot \tan \alpha_0}{(l+M/K_1S_1)} \left\{ \frac{v_0}{2^{1/2}} \frac{\cos(pt+\epsilon)}{(1+lK_1S_1/M)^{1/2}} - u_0 \sin(pt+\epsilon) \right\}$$

Thus the maximum values of the amplitudes of the harmonic terms that enter into η are proportional to

$$\frac{\tan \alpha_0}{q(l+M/K_1S_1)(1+lK_1S_1/M)^{1/2}} \text{ and } \frac{\tan \alpha_0}{q(l+M/K_1S_1)}, \quad . \quad (70)$$

i.e., inversely proportional to

$$\left(l+\frac{M}{K_1S_1}\right)\left(1+\frac{lK_1S_1}{M}\right)^{1/2}\left\{\frac{\kappa}{2} + \frac{\left(\frac{M}{K_1S_1}\right)^2 - \frac{l}{lK_2S_2}\left(1+\frac{K_2S_2}{K_1S_1}\right)}{V_0\left(l+\frac{M}{K_1S_1}\right)^2}\right\},$$

and

$$\left(l+\frac{M}{K_1S_1}\right)\left\{\frac{\kappa}{2} + \frac{\left(\frac{M}{K_1S_1}\right)^2 - \frac{l}{lK_2S_2}\left(1+\frac{K_2S_2}{K_1S_1}\right)}{V_0\left(l+\frac{M}{K_1S_1}\right)^2}\right\}, \quad (71)$$

respectively. This is also true of θ.

It is inadvisable to ignore the quantity κ in the expressions (71).

From the condition that (71) shall be as large as possible, it follows that *the tail should be fairly far behind the main plane, and the ratio area/load should be small. The margin of stability should be considerable.*

SUMMARY AND GENERAL CONCLUSIONS.

(20) In Part I., arts. 4-6, equations are found for determining the motion for some time after any given velocity is suddenly imparted to the air. In order to avoid too curved a path, the velocity of the aeroplane should be large, and the angle of attack and the ratio area/load both small. These conditions are limited by the consideration that it is useful to have diminishing curvature after the disturbance has set in, art. 8, and further by the necessity of avoiding great inclinations of the axis of the machine, art. 9. For this purpose it is also important to avoid too large a tail.

In Part II., the effect of an impulsive gust is shown to be equivalent to a velocity imparted to the air, and a rotation to the machine. Equations are found to determine the motion after the gust. The conditions found in Part I. also hold for Part II.

In Part III., the forced oscillations produced by a periodic gust are found. Both for an undamped and a damped gust, it is found that the conditions: large velocity, and small angle of attack and ratio area/load, prevent violent oscillations

in the path, arts. 15, 17. It is also advisable to give a considerable margin of stability, art. 16. Finally, in order to prevent a violent forced oscillation due to a damped wind agreeing in complex period with the slow free oscillation of the aeroplane, the ratio area/load should be small, and the tail should be at a considerable distance behind the main plane, the margin of stability also being appreciable.

In general we may conclude that from the point of view of this paper, *the ideal aeroplane is one that combines the following characteristics: large velocity, small angle of attack, small ratio area/load, small tail fairly far behind the main plane, and considerable margin of stability*

Practically it means that the heavy machine with a powerful engine at full speed is the safest.

AN ALIGNMENT CHART FOR OBTAINING HEIGHTS FROM OBSERVATIONS OF PRESSURE & TEMPERATURE.

BY A. H. STUART, B.SC., F.R.A.S.

The solution of Laplace's well known formula giving the relation between the height above datum level and the pressure and temperature of the atmosphere, may be rapidly accomplished by means of a d'Ocagne alignment chart, an example of which is shown in Fig. 1. Its use may be best explained by means of an example. Let the initial temperature (t_o) and pressure (p_o) be 20° C. and 29 inches of mercury, and at a certain height (z feet) suppose they have fallen to temperature (t) 0° C. and pressure (p) 20 inches. The ratio of the pressures $R = \frac{p_o}{p} = \frac{29}{20} = 1.45$, and the mean temperature $\frac{1}{2}(t_o + t) = 10$. Place a straight edge across the chart so as to cut the right hand (R) scale at 1.45 and the left hand (T) scale at 10. The cut of the middle (z) scale gives the required altitude, namely, 10,000 feet.

Below is given a brief description of the principle and method of construction of the chart.

If three scales are arranged, as shown in Fig. 2, it is easy to prove that

$$Z = Z_o + \frac{C}{A}X - \frac{B}{A}Y.$$

By using logarithmic scales d'Ocagne has made this fact of use in the solution of equations of the type $Z = \frac{KX^n}{Y^m}$. For $\log. Z = \log. K + n \log. X - m \log. Y$, which is an equation of exactly similar form.*

If p_o represent the pressure of air at a given point and p be the pressure at an altitude of **Z** feet above that point while the mean temperature over that range is T on the absolute scale, Laplace has shown that:—

$$\mathbf{Z} = 6.04 \times 10^4 \left(\frac{T}{273}\right) \log._{10} \frac{p_o}{p}$$

$$= 2.21 \times 10^2 \, T \log._{10} \frac{p_o}{p}$$

Let the ratio $\frac{p_o}{p}$ be denoted by R.

* Vide Traité de Monographie. Théorie des abaques. Applications pratiques. Paris, 1899.

Then

$$\mathbf{Z} = 2.21 \times 10^2 \frac{T}{\frac{1}{\log. R}}$$

$$\therefore \log. \mathbf{Z} = \log. (2.21 \times 10^2) + \log. T - \log. \left(\frac{1}{\log. R}\right)$$

Compare this equation with $Z = Z_0 + \frac{C}{A} X - \frac{B}{A} Y$. It will be observed that $\frac{C}{A} = \frac{B}{A}$ and referring to Fig. 2 it will be seen that they must necessarily be equal to $\frac{1}{2}$, whereas their value should be unity. To counteract this effect the scales corresponding to X and Y (viz., T and R) must be drawn twice the size of that corresponding to Z (viz., **Z**). Table I. shows the calculation of the temperature scale, Table II. refers to the **Z** scale, while Table III. indicates the best method by which the R scale may be constructed.

In laying the scales down it is convenient to construct the outer scales of T and R in any convenient position. Some datum point is then obtained on the middle scale by calculating the value for any given values of T and R and placing a straight edge across these values. Working from this point and with the aid of Table II., the **Z** scale may then be constructed.

TABLE I.

T.	log. T.	Difference.	Diff. × 2.
253 = − 20°C.	2.4031	0	0
263 = − 10°C.	2.4200	0.0169	0.034
273 = 0°C.	2.4362	0.0331	0.066
283 = 10°C.	2.4518	0.0487	0.097
293 = 20°C.	2.4669	0.0638	0.128
303 = 30°C.	2.4814	0.0783	0.157
313 = 40°C.	2.4955	0.0924	0.185

TABLE II.

Z in 1,000ft.	log. **Z**.	**Z** in 1,000ft.	log. **Z**.
1	0	10	1.000
1.5	0.1761	11	1.041
2	0.301	12	1.079
2.5	0.398	13	1.114
3	0.477	14	1.146
4	0.602	15	1.176
5	0.699	16	1.204
6	0.778	17	1.230
7	0.845	18	1.255
8	0.903	19	1.279
9	0.954	20	1.301

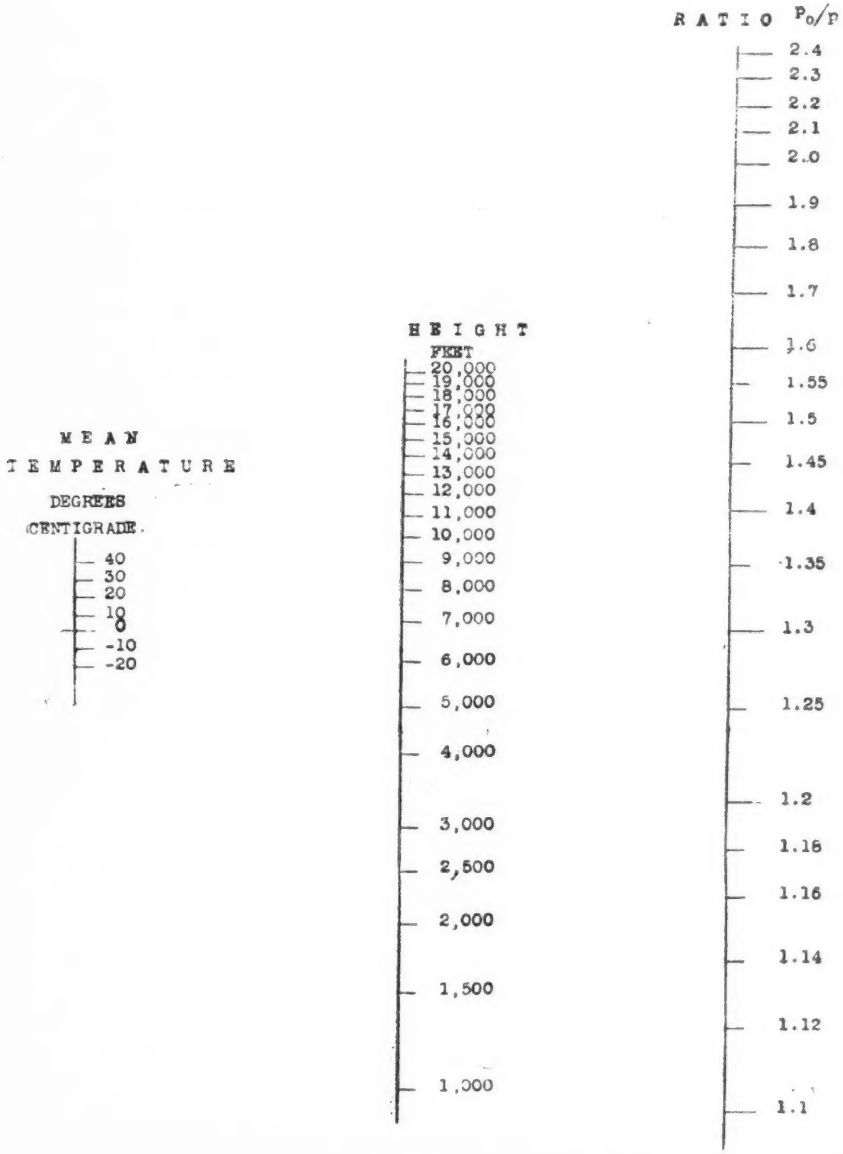

MEAN
TEMPERATURE
DEGREES
CENTIGRADE.
40
30
20
10
0
-10
-20
HEIGHT
FEET
20,000
19,000
18,000
17,000
16,000
15,000
14,000
13,000
12,000
11,000
10,000
9,000
8,000
7,000
6,000
5,000
4,000
3,000
2,500
2,000
1,500
1,000
RATIO P_0/P
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.55
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.16
1.16
1.14
1.12
1.1

FIG. 1.

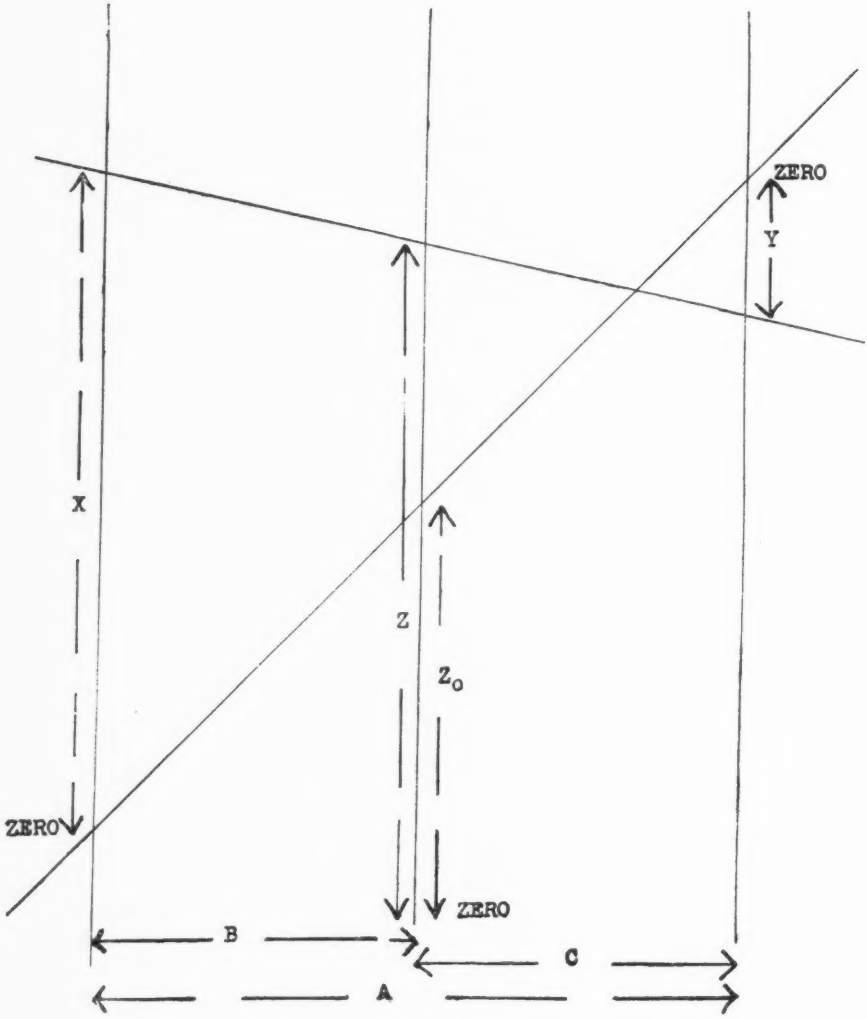


FIG. 2.

TABLE III.

$R.$	log. $R.$	$\frac{1}{\log. R.}$	$\log.\left(\frac{1}{\log. R}\right)$	Difference.	Diff. × 2.
1.1	0.0414	24.1	1.3820	0	0
1.12	0.0492	20.3	1.3075	0.0745	0.149
1.14	0.0569	17.6	1.2455	0.1365	0.273
1.16	0.0645	15.5	1.1903	0.1917	0.383
1.18	0.0719	13.9	1.1430	0.2390	0.478
1.2	0.0792	12.63	1.1014	0.2806	0.561
1.25	0.0969	10.3	1.0128	0.3692	0.738
1.3	0.1139	8.78	0.9435	0.4385	0.877
1.35	0.1303	7.68	0.8854	0.4966	0.993
1.4	0.1461	6.85	0.8357	0.5463	1.093
1.45	0.1614	6.20	0.7924	0.5896	1.179
1.5	0.1761	5.68	0.7543	0.6277	1.255
1.55	0.1903	5.26	0.7210	0.6610	1.322
1.6	0.2041	4.90	0.6902	0.6918	1.384
1.7	0.2304	4.34	0.6375	0.7445	1.489
1.8	0.2553	3.92	0.5933	0.7887	1.577
1.9	0.2788	3.59	0.5551	0.8269	1.654
2.0	0.3010	3.32	0.5211	0.8609	1.722
2.1	0.3222	3.10	0.4914	0.8906	1.781
2.2	0.3424	2.92	0.4654	0.9166	1.833
2.3	0.3617	2.765	0.4417	0.9403	1.881
2.4	0.3802	2.63	0.4200	0.9620	1.924

NOTE ON THE FLIGHT OF LOCUSTS.

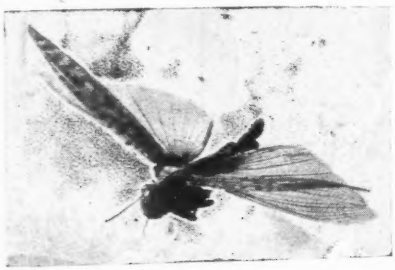
BY E. H. HANKIN, M.A., SC.D., A.F.AË.S., AGRA, INDIA.

In my article on the flight of locusts published in this journal (April-June, 1916, page 56) I described cases in which cheels were soaring and catching locusts in a light wind in which there were no irregularities of movement of the air having any visible effect on the flight of these insects.

In the following case, on the other hand, the movements of the locusts revealed irregularities in the movement of the air :—

> 23rd June, 1916, 1.30. At Jhansi Station. A light west wind and no sunshine. A small flight of locusts came in sight. These for the most part were flying directly up wind and making moderate speed over the ground. Crows were chasing and eating them. No cheels were visible. Probably the air was not soarable because there was no sunshine and because four cheels were seen in flap-gliding flight a few minutes later just outside Jhansi. Irregularities of the air were shown by the movement of the locusts. The distances from one to another often varied. Occasionally a mass of locusts moved relatively to other masses in a way that suggested the presence of local puffs of wind and in a way that did not resemble a change of course on the part of the insects.

Thus in one instance irregularities of air movement capable of affecting the flight of locusts were absent and such air was soarable. In the instance now brought forward such air irregularities were present and the air appears to have been unsoarable.



The accompanying illustration shows the appearance of a locust when in gliding flight. The photograph was made from a dried specimen. It will be seen that the fore wings are dihedrally up to a greater extent than the hind wings. The hind legs are tucked up against the sides of the abdomen and not trailing behind as when starting or stopping. So far as I have seen locusts always glide with a well marked loss of height. It may be one in four or one in five. At the moment of commencing a glide the locust appears to make a short dive. It descends for a fraction of a second a little more steeply than it does afterwards. There is an entry in my notes to the effect that there often appears to be a loss of speed during the glide. While in flapping flight the antennæ are held pointing out straight in front. I was able to see this several times on one occasion when a swarm of locusts were flying across the roof of my house and when the conditions of illumination were unusually good. The speed of flight was repeatedly measured on this occasion and found to be four metres per second.

REVIEWS.

The Petrol Engine. F. J. Kean. London, 1915: E. and F. N. Spon. pp. 124. Figs. Price 4s.

Aeronautical Engines. F. J. Kean. London, 1916: E. and F. N. Spon. pp. 94. Illus. and Plates. Price 6s. net.

The first of these two text books gives a very clear and concise account of the elements of the petrol engine, and such a book should be in the hands of every pilot and air mechanic who has not already gained professional knowledge of the subject, or who is not familiar with ordinary workshop practice.

Chapters II. to VII. cover the mechanical arrangements. The principal parts are shown in diagrammatic drawings and their function described with a rather happy simplicity.

Chapter X. on two-stroke engines, and in particular on the author's own designs, are worthy of the specialist's consideration.

Chapters XI. and XII. on horse-power and fuel, introduce the reader to the elements of design as far as cylinder volume, speed, and fuel consumption are concerned.

The appendix covers some common troubles.

This book serves as introduction to the more specialised text book on "Aeronautical Engines."

Generally speaking, very low weight per horse-power is the fundamental aim of the designer. Good fuel and oil consumption is equally important. Reliability in an aeronautical engine has a meaning in aviation fundamentally different from its meaning in car practice.

Freedom from breakdown during flight is vitally important, but cost of construction, overhaul, upkeep and replacements is of quite secondary importance.

In Chapters I.-VII. the author discusses admirably the requirements of aeroplane motors and shows how these special needs lead to different types.

Evenness of firing and balancing are of extreme importance and the influence of these requirements is discussed.

The beginner will have some difficulty in giving a physical meaning to the balancing diagrams, but the professional petrol engineer will find them of interest.

Chapter VIII. on existing aviation motors is inevitably not up-to-date, but forms a very suitable introduction to Burls's more elaborate treatise.

Air Screws. M. A. S. Riach. London, 1916: Crosby, Lockwood and Son. pp. 128. Plate and Figs. Price 10s. 6d.

Drzewiecki (pronounced Jevetsky) was the first to draw the analogy between "foils" or shaped surfaces of right cylindrical form and elements of screw blades.*

On the application of screws to aviation Drzewiecki applied his method of

* In the AERONAUTICAL JOURNAL for January, 1914. Low gives Soreau as the authority for placing Drzewiecki's original papers in 1882. Apparently this is a mistake for 1892.—EDITOR.

water screw analysis to air screws. This method and the analogy on which it was based were conveniently summarised in this journal.*

Two logical developments have followed from the adoption of the analogy. Firstly, wind tunnel experiments have been carried out at the N.P.L. (Bramwell) which have made public for the first time reliable experimental evidence for the close approximation of the theory with practice. Secondly, methods of analysing and integrating over blades of various forms have been developed to a large extent, notably in the book under consideration.

To engineers with sound mathematical training it is usually satisfactory to obtain integrable expressions for simple standard cases, and proceed to more complex cases where graphical or statistical integration is required. The book is weak on the application of graphical and statistical methods to obtain approximate integration of expressions for thrust and torque when the variation of chord angle of attack at different ratios of translation and rotation (V/n) render the expressions too complex for integration.

Chapter VIII. deals with this latter side of calculation and might be developed largely with advantage to air screw designers.

The author does well to raise the questions of aspect ratio and indraught on which further experimental work is much required.

With regard to indraught, Eiffel in his "Nouvelles Recherches," p. 334, publishes a graphical representation of the velocity of indraught in front of an airscrew.

Calling the aeroplane velocity V and the indraught velocity v, the relative wind at the moment of striking the blade is $V+v$. The angle of the blade path with the line of flight increases from $\tan^{-1} V/\omega r$ to $\tan^{-1} V+v/\omega r$ and the chord angle of attack is reduced by the quantity $\tan^{-1} V+v/\omega r - \tan^{-1} V/\omega r$.

At the same time the relative wind velocity squared is increased from $V^2+\omega^2 r^2$ to $(V+v)^2+\omega^2 r^2$. When ωr is about five times V, as is usually the case, the decrease of the angle of attack is much more important than the increase of velocity squared. So that an airscrew designed without this correction would "race."

On the question of aspect ratio, it may be assumed that there is no end leakage as in aerofoils. And probably it would be nearer the mark to assume infinite aspect ratio for each element of the blade. These two further approximations bring theory nearer to test results. The special boss and tip eddy losses must be investigated separately.

The question of aspect ratio in blades presents differences from that in aerofoils which may possibly prove fundamental.

On page 34 the name "Normale" should be ascribed to Drzewiecki.

The author is to be commended for adopting the official terminology of the Aëronautical Society.

A.

The Aeroplane. A. Fage. London, 1915: Charles Griffin and Co. pp. 136. Illus. Price 6s.

It is convenient to take Chapter VII., "The Aerial Propeller," as Mr. A. Fage calls the air screw, in proximity to the review of Mr. Riach's book.

He insists on giving to F. W. Lanchester a priority to which Drzewiecki is apparently entitled (p. 89). Drzewiecki's analogy with aerofoils is accepted by the

* "Airscrews," A. R. Low, Apr., 1913.

author as a sound approximate experimental basis for calculating the torque and thrust of the air screw as a whole. On page 97 the reference to "frictional contact" of the air with the blades is unnecessary, as this analogy of course includes frictional reaction in the test curve of the drag coefficient for the analogous aerofoil.

The range of efficiency 60%-75% in common practice may seem small, but with increase of efficiency the limiting size or performance increases as the square, and if we can get 75% efficiency by calculation it is poor engineering to be content with 60%.

Most of the now standard calculations are briefly but well treated, and the statistical calculation might serve as a model for completing Mr. Riach's more elaborate treatment. The reviewer has not yet seen published a correct account of the process by which maximum efficiency is obtained, given the r.p.m. and horse-power absorbed.

The rest of the book deals with aeroplanes, except the initial chapter on words and the final chapter on engines. The treatment throughout is exceptionally good and places this book at the head of textbooks on the subject.

It should be on the bookshelf of every designer, pilot, and mechanic.

Chapter VI. brings the very difficult subject of aeroplane stability for the first time within the reach of the technical engineer.

A quantitative idea of the phenomena has long been desired by aeroplane designers and pilots.

This chapter brings us much nearer the day when the behaviour of an aeroplane in gusty weather will be estimated by numerical coefficients rather than by the pilot's temperament.

A.

First Annual Report of the National Advisory Committee for Aeronautics. American National Advisory Committee for Aeronautics. Washington, U.S.A., 1916: Government Printing Office. pp. 303. Plates and Figs.

Our American cousins have paid our Advisory Committee and N.P.L. the high compliment of close imitation. It is hoped to deal in a later issue with the great mass of valuable material collected here.

But it is worth saying here and now that car engine designers and manufacturers will avoid many false steps by studying report No. 7, which gives by far the most elaborate analysis we have seen of existing aeroplane engines and the special developments they have followed.

NEW BOOKS AND PUBLICATIONS.

The Pictorial Russian Course. Russian in 30 Lessons. S. J. Luboff. London, 1916: L. M. Hill. pp. 141. Illus. Price 2s. net.

"Aerial Russia." The Romance of the Giant Aeroplane. Lieut.-Colonel B. Roustam Bek. London, 1916: John Lane, the Bodley Head. pp. 154. Illus. Price 2s. 6d. net.

On the Representation of an Aeroplane by Equivalent Surfaces in Mutually Orthogonal Planes. S. Brodetsky. Sendai, Japan, June, 1916: Tohoku Imperial University. Figs.

Report of the Advisory Committee for Aeronautics for the Year 1915-1916. London, 1916: Sir Joseph Causton and Sons, Ltd. pp. 12. Price 1d.

Imperial College of Science and Technology (Session 1916-1917). London, 1916: Eyre and Spottiswoode. pp. 457. Price 6d.

Report No. 8, General Specifications Covering Requirements of Aeronautic Instruments. American National Advisory Committee for Aeronautics. Washington, U.S.A., 1916: Government Printing Office. pp. 6.

Dynamical Stability of Aeroplanes. J. C. Hunsaker. Washington, U.S.A., 1916: The Smithsonian Institution. pp. 78. Plates and Figs.

The Aviation Pocket Book. R. Borlase Matthews. London, 1916: Crosby Lockwood and Son. pp. 176. Figs. Price 3s. 6d. net.

Flying Men and Their Machines. Clarence Winchester. London, 1916: J. M. Dent and Sons, Ltd. pp. 242. Illus. Price 6s. net.

The Computer's Handbook, Meteorological Office, Introduction to. London, 1916: Darling and Son, Ltd. pp. 52. Price 1s.

"With Wings Outspread." Marc Gouvrieux. London, 1916: William Heinemann. pp. 243. Price 5s. net.

Aeronautical Engines. F. J. Kean. London, 1916: E. and F. N. Spon. pp. 94. Illus. and Plates. Price 6s. net.

"Eclipse or Empire." H. B. Gray and Samuel Turner. London, 1916: Nisbet and Co., Ltd. pp. 316. 1 Plate. Price 2s. net.